The average order of the Dirichlet series of the gcd-sum function

Kevin A. Broughan

*University of Waikato, Hamilton, New Zealand*

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E-mail: kab@waikato.ac.nz

The second term and improved error expressions for the partial sums of the
Dirichlet series of the gcd-sum function, for all real values of the parameter,
are derived using a result of Bordelles.

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Concerned with sequence A018804.

1. **INTRODUCTION**

The second term and improved error expressions for the partial sums of
the Dirichlet series of the gcd-sum function, first given in [2], are derived,
for all real values of the parameter, using a result of Bordelles [1].

Let \((m, n)\) represent the positive greatest common divisor of the positive
integers \(m\) and \(n\), let \(g(n) := \sum_{i=1}^{n}(i, n)\) be the gcd-sum, let \(\tau(n)\) be the
number of divisors of \(n\), let \(Id(n) := n\) be the identity function, for \(\alpha \in \mathbb{R}\)
let \(m_{\alpha}(n) = 1/n^\alpha\) be the monomial power function, and let \(\phi(n)\) be Euler’s
phi function. For \(x \geq 1\) let

\[
G_\alpha(x) := \sum_{n \leq x} \frac{g(n)}{n^\alpha}
\]

be the average order.

Let \(\theta\) be the smallest positive real number such that

\[
\sum_{n \leq x} \tau(n) = x \log x + x(2\gamma - 1) + O_{\epsilon}(x^{\theta+\epsilon}) \quad (1)
\]

for all \(\epsilon > 0\) and \(x \geq 1\). For a discussion about the value of \(\theta\) and references
see [1]. The current best value is slightly more than 1/4.
The Lemma of Bordelles [1] enables us to write $g = \mu * (\tau \cdot Id)$ and this, combined with a best available error for the average order of $\tau(n)$, enables a further term and greater degree of precision to be obtained for the $G_\alpha(x)$ than the expression $g = \phi * Id$ used in [2]. This is true for every value of $\alpha$ except for $\alpha = 2$ in which case both expressions for $g(n)$ give the same terms and error form.

In Section 3 we correct some errors in [2].

2. AVERAGE VALUES OF THE DIRICHLET SERIES

First we give two summations for the sum of divisor function $\tau(n)$.

**Lemma 2.1.** For all $\beta$ satisfying $\beta \geq -\theta$ and $\epsilon > 0$:

$$\sum_{n \leq x} n^{\beta} \tau(n) = \frac{x^{\beta+1}}{\beta+1} \log x + \frac{x^{\beta+1}}{\beta+1} (2\gamma - \frac{1}{\beta+1}) + O_\epsilon(x^{\beta+\epsilon}).$$

*Proof.* This follows from (1) and Abel summation as in Lemma 1 of [1].

**Lemma 2.2.** For all $\beta$ satisfying $-1 < \beta < -\theta$:

$$\sum_{n \leq x} n^{\beta} \tau(n) = \frac{x^{\beta+1}}{\beta+1} \log x + \frac{x^{\beta+1}}{\beta+1} (2\gamma - \frac{1}{\beta+1}) + O(1)$$

where the implied constant is absolute.

Let $c_1 := 2\gamma - \frac{1}{2} - \frac{\zeta'(2)}{\zeta(2)}$. The theorem of Bordelles [1] can be written:

**Lemma 2.3.** For all $\epsilon > 0$, as $x \to \infty$

$$G_0(x) = \frac{x^2 \log x}{2\zeta(2)} + c_1 \frac{x^2}{2\zeta(2)} + O(x^{1+\theta+\epsilon}).$$

It is not needed in what follows, but for completeness the corresponding sum for the power of the multiplier -1 is [3]:

$$\sum_{n \leq x} \frac{\tau(n)}{n} = \frac{\log^2 x}{2} + 2\gamma \log x + \gamma^2 - 2\gamma_1 + O(\frac{1}{\sqrt{x}}),$$

where $\gamma_1$ is the first Stieltjes constant.

Figure 1 is a plot of $G_0(x) - \frac{x^2 \log x}{2\zeta(2)} - c_1 \frac{x^2}{2\zeta(2)}$ for $1 \leq x \leq 1000$. This should be compared with the corresponding plots for $G_\alpha$ given below. It
FIG. 1. Error value for $G_0(x)$.

shows that, maybe, another continuous term can be extracted from the error. The existing computed error is large, but does, on the face of it, go to zero when divided by $x^2$, consistent with the given value of the constant $c_1$.

We can now improve each part of [2, Theorem 4.4]:

THEOREM 2.1. For all $\epsilon > 0$ as $x \to \infty$

(1) If $\alpha \leq 1 + \theta$:

$$G_\alpha(x) = \frac{x^{2-\alpha} \log x}{(2 - \alpha)\zeta(2)} + \frac{x^{2-\alpha}}{(2 - \alpha)\zeta(2)} (2\gamma - \frac{1}{2 - \alpha} - \frac{\zeta'(2)}{\zeta(2)}) + O(x^{\theta+1-\alpha+\epsilon}).$$

(2) If $1 + \theta < \alpha < 2$:

$$G_\alpha(x) = \frac{x^{2-\alpha} \log x}{(2 - \alpha)\zeta(2)} + \frac{x^{2-\alpha}}{(2 - \alpha)\zeta(2)} (2\gamma - \frac{1}{2 - \alpha} - \frac{\zeta'(2)}{\zeta(2)}) + O(1).$$

(3) $G_2(x) = \frac{\log^2 x}{2\zeta(2)} + \frac{\log x}{\zeta(2)} (2\gamma - \frac{\zeta'(2)}{\zeta(2)}) + O(1).$
(4) If \( \alpha > 2 \):

\[
G_\alpha(x) = \frac{\zeta(\alpha - 1)^2}{\zeta(\alpha)} - \frac{\log x}{(\alpha - 2)\zeta(2)x^{\alpha - 2}}
\]

\[
= \frac{1}{(\alpha - 2)\zeta(2)x^{\alpha - 2}} \left( 2\gamma + \frac{1}{\alpha - 2} - \frac{\zeta'(2)}{\zeta(2)} \right) + O_x(\frac{1}{x^{\alpha - 1 - \epsilon}}).
\]

**Proof.**

(1) In this case \( \alpha \leq 1 + \theta \). Let \( \beta := 1 - \alpha, c_2 := 2\gamma - 1/(1 + \beta) \). Then, using Lemma 2.1 and the complete multiplicativity of \( m_\alpha \):

\[
G_\alpha(x) = \sum_{n \leq x} \frac{g(n)}{n^\alpha} = \sum_{n \leq x} (m_\alpha \cdot g)(n)
\]

\[
= \sum_{n \leq x} ((m_\alpha \cdot \mu) * (m_\alpha \cdot Id \cdot \tau))(n)
\]

\[
= \sum_{d \leq x} \frac{\mu(d)}{d^\alpha} \sum_{e \leq x/d} e^{1-\alpha} \tau(e)
\]

\[
= \sum_{d \leq x} \frac{\mu(d)}{d^\alpha} \left[ \frac{1}{\beta + 1} \frac{x^{\beta + 1}}{d^{\beta + 1}} \log x - \log d \right] + c_2 \frac{x^{\beta + 1}}{d^{\beta + 1}} + O_x(\frac{x^{\beta + \epsilon}}{d^{\beta + 1}})
\]

\[
= \frac{x^{\beta + 1} \log x}{\beta + 1} \sum_{d \leq x} \frac{\mu(d)}{d^{\alpha + \beta + 1}} - \frac{x^{\beta + 1}}{\beta + 1} \sum_{d \leq x} \frac{\mu(d) \log d}{d^{\alpha + \beta + 1}}
\]

\[
+ c_2 x^{\beta + 1} \sum_{d \leq x} \frac{\mu(d)}{d^{\alpha + \beta + 1}} + O_x(\frac{x^{\beta + \epsilon}}{d^{\beta + 1}})
\]

Therefore

\[
G_\alpha(x) = \frac{x^{2 - \alpha} \log x}{2 - \alpha} \left[ \frac{1}{\zeta(2)} - \sum_{d > x} \frac{\mu(d)}{d^2} \right] - \frac{x^{2 - \alpha}}{2 - \alpha} \left[ \frac{\zeta'(2)}{2 - \alpha} - \sum_{d > x} \frac{\mu(d) \log d}{d^2} \right]
\]

\[
+ c_2 x^{2 - \alpha} \left[ \frac{1}{\zeta(2)} - \sum_{d > x} \frac{\mu(d)}{d^2} \right] + O_x(\frac{x^{\beta + \epsilon + 1 - \alpha}}{d^{\beta + 1}})
\]

\[
= \frac{x^{2 - \alpha} \log x}{(2 - \alpha)\zeta(2)} + \frac{x^{2 - \alpha}}{(2 - \alpha)\zeta(2)} \left[ 2\gamma - \frac{1}{2 - \alpha} - \frac{\zeta'(2)}{\zeta(2)} \right]
\]

\[
+ O_x(\frac{x^{\beta + \epsilon + 1 - \alpha}}{d^{\beta + 1}})
\]
(2) The derivation is the same as in (1), except we use Lemma 2.2 instead of Lemma 2.1.

(3) This is given in the proof of [2, Theorem 4.4, Case 2].

(4) Let \( \alpha > 2 \). Then, using [2, Theorem 4.1] and Abel summation:

\[
G_{\alpha}(x) = \frac{\zeta(\alpha - 1)^2}{\zeta(\alpha)} - \sum_{n>x} \frac{g(n)}{n^\alpha} \\
= \frac{\zeta(\alpha - 1)^2}{\zeta(\alpha)} - \lim_{y \to \infty} \frac{G_0(y)}{y^\alpha} + \frac{G_0(x)}{x^\alpha} - \alpha \int_x^\infty \frac{G_0(t)}{t^{\alpha+1}} dt \\
= \frac{\zeta(\alpha - 1)^2}{\zeta(\alpha)} + \frac{G_0(x)}{x^\alpha} - \alpha \int_x^\infty \frac{G_0(t)}{t^{\alpha+1}} dt \\
= \frac{\zeta(\alpha - 1)^2}{\zeta(\alpha)} - \frac{\log x}{(\alpha - 2)\zeta(2)x^{\alpha-2}} \\
\quad - \frac{1}{2(\alpha - 2)\zeta(2)x^{\alpha-2}} [2c_1 + \frac{\alpha}{\alpha - 2}] + O\left(\frac{1}{x^{\alpha-1-\theta-\epsilon}}\right),
\]

and the result follows after substituting for \( c_1 \).

\( \quad \)

**FIG. 2.** Error value for \( G_{\alpha}(x) \) with \( \alpha = 1 \).

Figure 2 is a plot of \( G_1(x) - \frac{x \log x}{\zeta(2)} - \frac{x}{\zeta(2)} (2\gamma - 1 - \frac{\zeta'(2)}{\zeta(2)}) \) for \( 1 \leq x \leq 1000 \).
Figure 3 is a plot of

\[ G_\alpha(x) = \frac{x^{2-\alpha} \log x}{(2 - \alpha)\zeta(2)} - \frac{x^{2-\alpha}}{(2 - \alpha)\zeta(2)}(2\gamma - \frac{1}{2 - \alpha} - \frac{\zeta'(2)}{\zeta(2)}) \]

for \( \alpha = 3/2 \) and \( 1 \leq x \leq 1000. \)

3. ERRORS IN “THE GCD-SUM FUNCTION”

Corrections to some errors in [2]. Page 7 line 8: the second \( g \) should be \( Id \), page 9 line 3: 5.1 should be 4.1, page 12 line 2: 5.4 should be 4.3, page 12 line 3: 5.3 should be 4.2, page 13 line 13 and line -1: \( e, d \) should be \( e.d \), page 14 line -1: 5.4 should be 4.4.

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REFERENCES

2. Broughan, K.A. *The gcd-sum function*, Journal of Integer Sequences, **4** (2001), Article 01.2.2.